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NONLINEAR STABILITY ANALYSIS OF A UNIFORMLY FLUIDIZED BED

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Abstract—A small amplitude long wavelength nonlinear analysis is used to examine the stability of a uniformly fluidized bed. This work has the same objectives as that of a previous study by Liu. Although both analyses are based on the same model, there are major differences in interpretation and in the formula for the equilibrium amplitude. The analysis presented here implies that small amplitude sinusoidal perturbations about an unstable state do not equilibrate in a gas fluidized bed, in contrast to Liu's conclusion that there is equilibration. Numerical calculations on the original model describing a gas fluidized bed verify the analysis given here and extend it to larger amplitudes and smaller wavelengths.

Key Words: nonlinear analysis, fluidization, stability, periodic, traveling waves

1. INTRODUCTION

In the past, linearized equations based on the two-fluid model have been used to investigate the stability of the state of uniform fluidization (e.g. Pigford & Baron 1965; Anderson & Jackson 1968; Needham & Merkin 1983; Liu 1982). The studies have shown that, in many instances, these equilibrium states are unstable. It is of course taken for granted that the perturbations must be of a relatively small amplitude in order for the analysis to be consistent. In the case when the perturbations grow in amplitude, it is clear that nonlinear considerations become important.

Many nonlinear analyses have been performed on the equations of motion (e.g. Fanucci *et al.* 1979, 1981; Needham & Merkin 1983; Liu 1983). However, the relationship between the linear stability analysis and the nonlinear analysis is studied in detail only in the work of Liu (1982, 1983). The nonlinear analysis of Needham & Merkin (1983, p. 449) only briefly mentions that a limit cycle bifurcates from an equilibrium point in an equation that describes traveling wave solutions of the model equations. This periodic solution is neither calculated nor its relationship to linearly unstable perturbations discussed.

The analysis presented here has the same objectives as those of Liu (1983): a small amplitude, long wavelength, nonlinear study about the uniform state which leads to a "Landau" equation for the amplitude. Both studies are based on the same model equations and make the same simplifying assumptions (although at different places in the analysis). In both studies the original system of partial differential equations is replaced with a simpler nonlinear equation for one dependent variable. Ostensibly, the equations are similar. However, there are important differences in sign as well as in the interpretations of the effects of the nonlinearities on the evolution of the perturbations. Liu's perturbation analysis of his simplified equation follows closely the work of Stuart (1960) and results in an evolution for the amplitude of the unstable disturbance. We adopt a different scheme, using instead a multiple time scale analysis employed by Ganser (1983) on a very similar problem. This also leads to an evolution equation for the amplitude which has a different (nontrivial) equilibrium amplitude. The conclusions of Liu and those given in this paper are in opposition to one another. Liu's calculations imply equilibration of small amplitude and long wavelength disturbances (in a gas fluidized bed), while our calculations show there is no equilibration and that the equilibrium amplitude is unstable. Our analysis is substantiated by evidence obtained from numerical experimentation.

In both studies, the analysis is limited to spatial variations in only the vertical direction. We restrict considerations to gas fluidized particles and neglect the inertia of the gas as well as virtual mass effects. However, we do include particle viscosity and particle pressure and assume that they are functions of particle concentration. This is basically the model studied by Needham & Merkin (1983) and Liu (1983). Liu initially includes the inertia of the fluid phase and virtual mass effects. However, he later assumes the ratio of the fluid density to particle density to be small and thus his model effectively reduces to the simpler model.

Section 2 briefly presents the general two-fluid model and its simplifications for gas fluidized particles. A review of the relevant results of linear theory is given in section 3 and, in particular, the existence of a critical wave number that separates stable wave numbers (larger) from unstable wave numbers (smaller). Section 4 contains the nonlinear analysis related to the existence and temporal stability of the periodic solutions that bifurcate from the uniformly fluidized state at the critical wave number. In section 5 numerical calculations of the periodic solutions show agreement kwith the results of section 4 for small amplitude and large wavelength (small wave number) and the section also covers the case of larger amplitudes and smaller wavelengths.

2. BASIC EQUATIONS

In what follows, the subscript d will refer to the particle or dispersed phase and the subscript c to the fluid or continuous phase. We assume each phase is incompressible. The conservation of mass equations restricted to one-dimension with the positive x-axis pointing upwards are

$$\frac{\partial \alpha}{\partial t} + \frac{\partial(\alpha v_d)}{\partial x} = 0 \qquad [1a]$$

and

$$\frac{\partial(1-\alpha)}{\partial t} + \frac{\partial((1-\alpha)v_c)}{\partial x} = 0, \qquad [1b]$$

where α is the dispersed phase or particle volume concentration and v is velocity. The conservation of momentum equations for both phases are (Drew 1983; Homsy *et al.* 1980)

$$\rho_d\left[\frac{\partial(\alpha v_d)}{\partial t} + \frac{\partial(\alpha v_d^2)}{\partial x}\right] = \frac{\partial(T_d)}{\partial x} - I - \rho_d \alpha g \qquad [2a]$$

and

$$\rho_d\left[\frac{\partial((1-\alpha)v_c)}{\partial t} + \frac{\partial((1-\alpha)v_c^2)}{\partial x}\right] = \frac{\partial(T_c)}{\partial x} + I - \rho_c(1-\alpha)g, \qquad [2b]$$

where

$$I = -p_c\frac{\partial \alpha}{\partial x} - B(v_c - v_d) - \alpha\rho_c C_{VM}\left[\left(\frac{\partial(v_c)}{\partial t} + v_c\frac{\partial(v_c)}{\partial x}\right) - \left(\frac{\partial(v_d)}{\partial t} + v_d\frac{\partial(v_d)}{\partial x}\right)\right],$$

$$T_d = -p_d\alpha + \alpha\mu_d\frac{\partial(v_d)}{\partial x}$$

and

$$T_c = -p_c(1-\alpha) + (1-\alpha)\mu_c\frac{\partial(v_c)}{\partial x}.$$

The drag coefficient is denoted by B, C_{VM} is the virtual mass coefficient, p is the pressure and μ is the viscosity of each phase. As is frequently done in analytical work, a linear drag law is assumed (Needham & Merkin 1983; Liu 1982, 1983; Fanucci *et al.* 1979, 1981). The form for the virtual mass effect is chosen to be that used by Liu (1983). Substitution of the forms for I, T_d and T_c into [2a, b] yields

$$\rho_d\left[\frac{\partial(\alpha v_d)}{\partial t} + \frac{\partial(\alpha v_d^2)}{\partial x}\right] = -\alpha\frac{\partial(p_c)}{\partial x} - \frac{\partial(\alpha(p_d - p_c))}{\partial x} - \alpha\rho_d g + B(v_c - v_d) + \frac{\partial\left(\alpha\mu_d\frac{\partial v_d}{\partial x}\right)}{\partial x}$$
$$+ \alpha\rho_c C_{VM}\left[\left(\frac{\partial(v_c)}{\partial t} + v_c\frac{\partial(v_c)}{\partial x}\right) - \left(\frac{\partial(v_d)}{\partial t} + v_d\frac{\partial(v_d)}{\partial x}\right)\right] \qquad [3a]$$

and

$$\rho_c\left[\frac{\partial((1-\alpha)v_c)}{\partial t}+\frac{\partial((1-\alpha)v_c^2)}{\partial x}\right]=-(1-\alpha)\frac{\partial(p_c)}{\partial x}-\rho_c(1-\alpha)g-B(v_c-v_d)$$

$$+\frac{\partial\left((1-\alpha)\mu_c\frac{\partial(v_c)}{\partial x}\right)}{\partial x}-\alpha\rho_c C_{VM}\left[\left(\frac{\partial(v_c)}{\partial t}+v_c\frac{\partial(v_c)}{\partial x}\right)-\left(\frac{\partial(v_d)}{\partial t}+v_d\frac{\partial(v_d)}{\partial x}\right)\right]. \quad [3b]$$

Since B is assumed to be only a function of α, this dependence can be determined by steady-state experiments. A typical form is (Anderson & Jackson 1968; Needham & Merkin 1983)

$$B(\alpha)=\alpha(1-\alpha)^{2-n}K,$$

where $K=9\mu_c/2R^2$, μ_c is the viscosity of the continuous phase and R is the radius of the particles. For our purposes a more convenient form is $K=g\rho_d/v_t$, where v_t is the terminal velocity of an isolated particle. Typical values for n are approx. 2–4 (Anderson & Jackson 1968; Homsy *et al.* 1980; Needham & Merkin 1983).

The magnitude of μ_d suggested by the work of Homsy *et al.* (1980) (liquid–solid), Needham & Merkin (1983) (gas–solid), Fanucci *et al.* (1981) (gas–solid) and Anderson & Jackson (1968) (both liquid–solid and gas–solid) is 10 P. Homsy *et al.* (1980) also estimate C_{VM} to be of the order of 2–8. The collisional pressure $\alpha(p_d-p_c)\equiv p_d v_t^2 F(\alpha)$ is assumed to be a function of α, as in the work of Homsy *et al.* (1980), Liu (1983), Drew (1983) and Needham & Merkin (1983). A similar constitutive assumption is made in the work of Foscolo & Gibilaro (1987). The derivative, $F'>0$ is further assumed constant, as in Homsy *et al.* (1980), Liu (1983) and Needham & Merkin (1983).

The analysis will be restricted to gas fluidized particles so that the inertia and viscosity of the gas phase can be neglected. This corresponds to setting $\rho_c=0$ in [3a, b]. This simplification allows p to be eliminated from [3a], yielding

$$\alpha\rho_d\left[\frac{\partial(v_d)}{\partial t}+v_d\frac{\partial(v_d)}{\partial x}\right]=-\alpha\rho_d g+\frac{B(v_c-v_d)}{1-\alpha}-\frac{\partial(\rho_d v_t^2 F(\alpha))}{\partial x}+\frac{\partial\left(\alpha\mu_d\frac{\partial(v_d)}{\partial x}\right)}{\partial x}. \quad [4]$$

If we add [1a, b] and integrate, we get

$$\alpha v_d+(1-\alpha)v_c=j, \quad [5]$$

where j is the volumetric flux—assumed to be a constant parameter. Equations [1a], [4] and [5] are the equations studied by Needham & Merkin (1983) and essentially those studied by Liu (1983), who assumes $\rho_c \ll \rho_d$ (p. 338) later in his calculations.

3. REVIEW OF LINEAR THEORY

An objective of this paper is to investigate the stability of a uniformly fluidized state. Using either [4] and [5], or [3a, b] and [5], those equilibrium states corresponding to $v_d=0$ with no spatial variation in α or v_c are given by

$$(1-\alpha_0)v_{c0}=j \quad [6a]$$

and

$$v_{c0}=v_t(1-\alpha_0)^{n-1}. \quad [6b]$$

To study the stability of these states, we introduce the dimensionless variables

$$x^*=\frac{x-c_k t}{L}, \quad [7a]$$

$$t^*=\frac{tv_{c0}}{L}, \quad [7b]$$

$$\alpha(x, t) = \alpha_0 + a\tilde{\alpha}(x^*, t^*), \tag{7c}$$

$$v_d(x, t) = av_{c0}\,\tilde{v}_d(x^*, t^*) \tag{7d}$$

and

$$v_c(x, t) = v_{c0} + av_{c0}\,\tilde{v}_c(x^*, t^*). \tag{7e}$$

The parameter $c_k = v_{c0}\,\alpha_0\,n$ is the linear kinematic wave speed, while L and "a" are the length scale and amplitude scale, respectively. Substitution of [7a–e] into [1a], [4] and [5], and retaining only terms of $O(a)$, yields the linearized equations. These can be used to find an equation in only the variable $\tilde{\alpha}$. See, for example, Liu (1982) or Needham & Merkin (1983). See also the more recent linear analysis by Batchelor (1988). In the current notation this equation is (superscript $\sim$ dropped)

$$0 = \alpha_{t^*} - R_1^{-1}\,\alpha_{t^*x^*x^*} + R_2^{-1}\,\alpha_{x^*x^*x^*} + \nu\,(\alpha_{t^*t^*} + (c_{0+} + c_{0-})\,\alpha_{x^*t^*} + c_{0+}\,c_{0-}\,\alpha_{x^*x^*}), \tag{8}$$

where

$$R_1^{-1} = \frac{\alpha_0(1-\alpha_0)^2\,\mu_{d0}}{B_0L^2}, \quad R_2^{-1} = \frac{c_k}{v_{c0}}\,R_1^{-1},$$

$$c_{0\pm} = -\frac{c_k}{v_{c0}} \pm \frac{v_t}{v_{c0}}\,(F')^{1/2},$$

$$\nu = \frac{L_c}{L}, \quad L_c = \frac{v_{c0}^2(1-\alpha_0)}{g}, \quad B_0 = B[\alpha_0]$$

and

$$\mu_{d0} = \mu_d(\alpha_0).$$

Except for a factor of $(1-\alpha_0)$, ν is the Froude number of the system. It is the ratio of two length scales. The distance over which the particle concentration is varying is denoted by L, and L_c is approximately the distance particles need to travel to change (either increase or decrease) their velocity by $O(v_{c0})$ while the net force acting on the particles is of the same magnitude as the force due to gravity. In the present model, the forces acting on the particles (gravity, interfacial forces and pressure) are of this magnitude. If $L \gg L_c$, then L_c will roughly correspond to the distance the particles travel (since the particles are required to adjust their velocity by order v_{c0}) before they reach equilibrium.

In the classical linear analysis, solutions of the form

$$e^{i(x^* - \omega t^*)}$$

are substituted into [8]. Note that since x^* is normalized, $1/L$ plays the role of the wave number which, in this case, corresponds in physical variables to a wavelength of $2\pi L$. As shown in Needham & Merkin (1983) or Liu (1982), the α_0 state is linearly stable with respect to all wave numbers when

$$F' > n^2(1-\alpha_0)^{2n-2}\,\alpha_0^2 \qquad (c_{0+} > 0). \tag{9a}$$

The critical state α_0^u is defined by the equation

$$F' = n^2(1-\alpha_0^u)^{2n-2}\,(\alpha_0^u)^2 \qquad (c_{0+} = 0). \tag{9b}$$

The α_0 states are linearly unstable, but only with respect to certain wave numbers, when

$$F' < n^2(1-\alpha_0)^{2n-2}\,\alpha_0^2 \qquad (c_{0+} < 0). \tag{9c}$$

It is clear from these calculations that particle pressure at high concentrations (small j) stabilizes the uniformly fluidized state for $\alpha_0 > \alpha_0^u$. As j is increased above a critical flux j_c corresponding to α_0^u, the α_0 states less than α_0^u become unstable. However, only the wave numbers smaller than L_0^{-1} (which depends on α_0) are unstable, where L_0 is given implicitly by the equation

$$\frac{-R_{20}^{-1}}{1 + R_{10}^{-1}} = c_{0+}\,. \tag{10}$$

The subscript 0 indicates that R_1^{-1} and R_2^{-1} are evaluated at $L = L_0$. Although there is another critical state $\alpha_0^L < \alpha_0^u$ that satisfies [9a], it will not play an important role in this study.

The regions of stability and instability given in [9a–c] and [10] have an interpretation in terms of a wave-hierarchy. Equations [9a, b] imply that the α_0 states are stable when the lower order kinematic speed is bounded above and below by the higher order speeds ($\pm\sqrt{F'}$) and unstable when the kinematic speed is no longer within these bounds. If the dispersion in the problem is associated with an improved kinematic model, then it is possible to retain the correspondence between wave speeds and stability with the result that wave speeds, and hence stability, depend on the wave number [10]. See Whitham (1974) for a general discussion of these ideas and Liu (1982) for the application to fluidization.

4. NONLINEAR PROBLEM

Our goal is to investigate the bifurcation at the critical wavelength L_0 using perturbation methods. This is accomplished by first deriving a small amplitude and long wavelength approximation to [1a], [4] and [5] about the α_0 states close to α_0^u (see the appendix). A multiple time scale analysis is carried out in this section to analyze the evolution of sinusoidal perturbations with wave numbers near L_0^{-1}. This will give information on the existence of small amplitude periodic solutions with wave numbers near L_0^{-1} as well as the stability of these solutions. In section 5 the calculations for the periodic solutions are extended numerically beyond the small amplitude regime. These calculations for the system given by [1a], [4] and [5] are shown to be in agreement with the small amplitude analysis in this section. First however, we compare our nonlinear equation [A.14] with the analogous equation studied by Liu (1983).

A small amplitude long wavelength approximation to [1a], [3a, b] and [5] has previously been given by Liu (1983). Due to differences in the results, our calculations based on [1a], [4] and [5] leading to [A.14] (repeated here for convenience) are given in the appendix for comparison:

$$\frac{\partial \alpha}{\partial t^*} - 2Na\alpha\frac{\partial \alpha}{\partial x^*} + R_2^{-1}\frac{\partial^3 \alpha}{\partial x^{*3}} + \nu c_{0-}\frac{\partial\left((c_{0+} + \tilde{c}_{0+}\, a\alpha)\dfrac{\partial \alpha}{\partial x^*} + \dfrac{\partial \alpha}{\partial t^*}\right)}{\partial x^*} = 0. \qquad [A.14]$$

Although Liu's (1983) derivation (equation 3.1 in his paper) is based on [1a], [3a, b] and [5], the results should agree with [A.14] if ρ_c/ρ_d is set equal to zero and boundary layer effects are ignored. These are assumptions which Liu makes later in his calculations (pp. 338, 343). When the change in notation is taken into account, there is agreement except for the nonlinear correction to the lower order wave speed $-2Na\alpha$ ($a_3\epsilon$ in Liu's paper). The coefficient N in our model vanishes at $\alpha_0 = 2/(n+1)$ which is the inflection point for the particle flux, αv, in the lower order kinematic model. Consequently, N will be positive for α_0 states larger than 0.4 if $n = 4$ (Needham & Merkin 1983), for α_0 states larger than 0.52 if $n = 2.87$ and for α_0 states larger than 0.6 if $n = 2.34$ (El-Kaissy & Homsy 1976). Thus, for the α_0 states of interest, the effect of the nonlinear correction to the lower order wave speed is to increase this speed for decreasing amplitude of $a\alpha$ (increasing ϵ in Liu's notation). Although Liu does not derive his nonlinear coefficient a_3, he demonstrates that it is negative based on his formula. He therefore concludes that for the nonlinear correction, $a_3\epsilon$, the effect is to decrease the lower order wave speed for increasing ϵ. Hence, in Liu's model the effect of the nonlinearity is to restore the stability condition, whereas in our model its effect is to maintain the instability condition.

While there is agreement in the second order operator in our [A.14] and Liu's (1983) equation 3.1, there is disagreement in what should be considered the nonlinear correction to the higher order wave speed. As suggested by [A.14], we consider $\tilde{c}_{0+}a\alpha$ to be the nonlinear correction. Since $\tilde{c}_{0+} > 0$ (see [A.8]), this nonlinear effect causes the higher order wave speed along c_{0+} to decrease for decreasing values of $a\alpha$. Liu considers $c_3^2\epsilon$ [see Liu's (1983) equation 3.1 and the subsequent paragraph] as the nonlinear correction. Since c_3^2 has the dimensions of velocity squared, this cannot be added to c_1 and his conclusions based on that assumption are meaningless. We claim that a negative velocity must first be factored out of c_3^2 leaving a negative quantity, having units of velocity, as a nonlinear correction to c_1. This would have the effect of decreasing, and not increasing as Liu concludes, the propagation speed c_1 for increasing amplitude of ϵ (decreasing values of $a\alpha$) and therefore agrees with our own conclusions.

Based on our calculations ($N > 0$, $\tilde{c}_{0+} > 0$) the nonlinear effects tend to maintain the instability condition in the linearized sense. We will demonstrate both analytically and numerically that this indeed is the case and that the nonlinearities do not provide a mechanism for amplitude equilibration—at least in the near linear regime.

On the other hand, Liu reached the opposite conclusions based on $a_3 < 0$ and $c_3^2\epsilon$ being the nonlinear correction to c_1. Even if a_3 is the correct term, in view of Liu's misinterpretation of $c_3^2\epsilon$ and that in reality there is a competition between the two effects in his model, it would be reasonable to expect a condition for equilibration depending on the relative magnitudes of a_3 and c_3^2. However, Liu's (1983) calculation implies that the amplitude always equilibrates in the supercritical case.

The method of solution of the nonlinear equation used by Liu is based on the work of Stuart (1960). Here, we use a multiple time scale analysis. Since the results based on this analysis are also different from those of Liu, they are presented for comparison.

In the calculations that follow, [A.12] and [A.14] yield the same results to the order considered. Since the calculations are simpler, they will be demonstrated with [A.12]. The corresponding initial condition is

$$\alpha(x^*, t^* = 0) = \Gamma(e^{ix^*} + e^{-ix^*}) = 2\Gamma \cos(x^*), \qquad -\infty < x^* < \infty.$$

With $L = L_0$ and $\alpha_0 < \alpha_0^u$, the equilibrated solution with zero amplitude propagates. As L is changed slightly we initially expect solutions to propagate with finite but very small amplitudes for L sufficiently close to L_0. Hence, expecting "a" to be small relative to R_2^{-1} we introduce a new scaled time variable $\tilde{t} = R_2^{-1} t^*$. With this change of variable, [A.12] becomes

$$\alpha_{\tilde{t}} - 2N\epsilon\alpha\alpha_{x^*} + \alpha_{x^*x^*x^*} + \kappa\epsilon\left(\frac{c_{0+}c_{0-}}{R_2^{-1}}\alpha_{x^*x^*} - c_{0-}\,\alpha_{x^*x^*x^*x^*}\right) + \kappa\epsilon^2 W(\alpha\alpha_{x^*})_{x^*} = 0, \qquad [11]$$

where $W = c_{0-}(\tilde{c}_{0+} + 2N)$, $\epsilon = a/R_2^{-1} \ll 1$ and $\nu = \kappa\epsilon$. Thus, a/R_2^{-1} and ν are assumed to be of the same order.

Since $\epsilon \ll 1$, we look for solutions of the form

$$\alpha = \alpha_0(x^*, \tilde{t}_0, \tilde{t}_1, \ldots) + \epsilon\alpha_1(x^*, \tilde{t}_0, \tilde{t}_1, \ldots) + \cdots. \qquad [12]$$

In [12] $\tilde{t}_i = \epsilon^i \tilde{t}$, which represents the use of multiple time scales. Without this added structure to our solution, secular terms appear in the higher orders of a regular perturbation scheme. The added degrees of freedom allow us to eliminate these terms at each level in the calculations. As a result, an equation involving the partial derivative $\alpha_{\tilde{t}_i}$ occurs at the corresponding $O(\epsilon^i)$ level in the calculation. It is then necessary at the end to reconstitute the dependence on the original variable $\tilde{t}$.

Substitution of [12] into [11] yields the following sequence of equations:

$$L(\alpha_0) \equiv (\alpha_0)_{\tilde{t}_0} + (\alpha_0)_{x^*x^*x^*} = 0, \qquad [13a]$$

$$L(\alpha_1) = -(\alpha_0)_{\tilde{t}_1} + 2N(\alpha_0)(\alpha_0)_{x^*} - \kappa\left[\frac{c_{0+}c_{0-}}{R_2^{-1}}(\alpha_0)_{x^*x^*} - c_{0-}(\alpha_0)_{x^*x^*x^*x^*}\right], \qquad [13b]$$

$$L(\alpha_2) = -(\alpha_0)_{\tilde{t}_2} - (\alpha_1)_{\tilde{t}_1} + 2N(\alpha_1\alpha_0)_{x^*} - \kappa\left[\frac{c_{0+}c_{0-}}{R_2^{-1}}(\alpha_1)_{x^*x^*} - c_{0-}(\alpha_1)_{x^*x^*x^*x^*}\right]$$
$$- \kappa W(\alpha_0(\alpha_0)_{x^*})_{x^*} \qquad [13c]$$

and

$$L(\alpha_3) = -(\alpha_0)_{\tilde{t}_3} - (\alpha_1)_{\tilde{t}_2} - (\alpha_2)_{\tilde{t}_1} + 2N[(\alpha_0\alpha_2)_{x^*} + \alpha_1(\alpha_1)_{x^*}]$$
$$- \kappa\left[\frac{c_{0+}c_{0-}}{R_2^{-1}}(\alpha_2)_{x^*x^*} - c_{0-}(\alpha_2)_{x^*x^*x^*x^*}\right] - \kappa W[(\alpha_0(\alpha_1)_{x^*})_{x^*} + (\alpha_1(\alpha_0)_{x^*})_{x^*}]. \qquad [13d]$$

The solution of [13a] satisfying the initial condition is

$$\alpha_0 = A_0(\tilde{t}_1, \tilde{t}_2, \ldots)\exp[i(x^* + \tilde{t}_0)] + \text{c.c.}, \qquad A_0(0, 0, \ldots) = \Gamma, \qquad [14]$$

where c.c. denotes the complex conjugate. Elimination of the secular terms in [13b] requires

$$(A_0)_{\tilde{t}_1} = \kappa\left(\frac{c_{0+}c_{0-}}{R_2^{-1}} + c_{0-}\right)A_0. \qquad [15a]$$

Thus,

$$\alpha_1 = \left\{\frac{-NA_0^2}{3}\exp[2i(x^* + \tilde{t}_0)] + \text{c.c.}\right\} + \{B_0(\tilde{t}_1, \ldots)\exp[i(2x^* + 8\tilde{t}_0)] + \text{c.c.}\}, \qquad [15b]$$

where

$$B_0(0, \ldots) = \frac{N\Gamma^2}{3}.$$

At the next order, elimination of secular terms gives

$$(A_0)_{\tilde{t}_2} = \frac{-2N^2 A_0^* A_0^2 i}{3} \qquad [16a]$$

(here, superscript * signifies complex conjugate)

and

$$(B_0)_{\tilde{t}_1} = 4\kappa\left(\frac{c_{0+}c_{0-}}{R_2^{-1}} + 4c_{0-}\right)B_0. \qquad [16b]$$

Consequently,

$$\alpha_2 = \frac{iA_0^2\kappa}{18}\left(6W - 2N\frac{c_{0+}c_{0-}}{R_2^{-1}} - 14c_{0-}N\right)\exp[2i(x^* + \tilde{t}_0)] + \frac{NA_0^* B_0}{3}\exp[i(x^* + 7\tilde{t}_0)]$$
$$+ \frac{N^2 A_0^3}{12}\exp[3i(x^* + \tilde{t}_0)] - \frac{NA_0 B_0}{3}\exp[i(3x^* + 9\tilde{t}_0)] + A_1(\tilde{t}_1, \ldots)\exp[i(x^* + \tilde{t}_0)]$$
$$+ B_1(t_1, \ldots)\exp[i(2x^* + 8\tilde{t}_0)] + C_0(\tilde{t}_1, \ldots)\exp[i(3x^* + 27\tilde{t}_0)] + \text{c.c.} \qquad [16c]$$

with appropriate initial conditions for A_1, B_1 and C_0, so that

$$\alpha_2(x^*, 0, \ldots) = 0.$$

Removal of the secular terms at $O(\epsilon^3)$ gives

$$(A_0)_{\tilde{t}_3} = -(A_1)_{\tilde{t}_1} + \kappa\left(\frac{c_{0+}c_{0-}}{R_2^{-1}} + c_{0-}\right)A_1 - \frac{NA_0^* A_0^2}{9}\kappa\left(9W - 2N\frac{c_{0+}c_{0-}}{R_2^{-1}} - 14Nc_{0-}\right), \qquad [17a]$$

$$(B_0)_{\tilde{t}_2} = -(B_1)_{\tilde{t}_1} + 4\kappa\left(\frac{c_{0+}c_{0-}}{R_2^{-1}} + 4c_{0-}\right)B_1 \qquad [17b]$$

and

$$(C_0)_{\tilde{t}_1} = 9\kappa\left(\frac{c_{0+}c_{0-}}{R_2^{-1}} + 9c_{0-}\right)C_0. \qquad [17c]$$

Since higher order results will not be needed, α_3 is not given. Using

$$\mathrm{d}A_0/\mathrm{d}\tilde{t} = \epsilon(A_0)_{\tilde{t}_1} + \epsilon^2(A_0)_{\tilde{t}_2} + \epsilon^3(A_0)_{\tilde{t}_3} + O(\epsilon^4),$$

[15a], [16a] and [17a] imply that

$$\frac{\mathrm{d}M}{\mathrm{d}\tilde{t}} = \epsilon\kappa\left(\frac{c_{0+}c_{0-}}{R_2^{-1}} + c_{0-}\right)M - \epsilon^2\left(\frac{2N^2 M^* M^2 i}{3}\right)$$
$$- \epsilon^3\frac{\kappa N}{9}\left(9W - 2N\frac{c_{0+}c_{0-}}{R_2^{-1}} - 14Nc_{0-}\right)M^* M^2 + O(\epsilon^4),$$

where $M = A_0 + \epsilon^2 A_1$. If we let $M = |M|\mathrm{e}^{i\theta}$, it then follows that

$$\theta_{\tilde{t}} = -\epsilon^2\frac{2N^2|M|^2}{3} + O(\epsilon^4) \qquad [18a]$$

and

$$\frac{\mathrm{d}|M|}{\mathrm{d}t^*} = \epsilon\kappa(c_{0+}c_{0-} + R_2^{-1}c_{0-})|M| - R_2^{-1}\epsilon^3\frac{\kappa N}{9}\left(9W - 2N\frac{c_{0+}c_{0-}}{R_2^{-1}} - 14Nc_{0-}\right)|M|^3 + O(R_2^{-1}\epsilon^4). \qquad [18b]$$

If [18b] is to have a nonzero equilibrium value for $|M|$, it is necessary that

$$\frac{c_{0+}c_{0-}}{R_2^{-1}} + c_{0-} = O(\epsilon^2). \qquad [19]$$

Using this result to simplify [18b] and noting that $W = c_{0-}(\tilde{c}_{0+} + 2N)$, we have

$$\frac{\mathrm{d}|M|}{\mathrm{d}t^*} = \epsilon\kappa|M|c_{0-}[c_{0+} + R_2^{-1} - R_2^{-1}\epsilon^2|M|^2N(\tilde{c}_{0+} + \tfrac{2}{3}N)] + O(R_2^{-1}\epsilon^4). \qquad [20]$$

Equation [20] implies that the nontrivial equilibrium value for $|M_0|$ is given by the equation

$$2a|M_0| = \frac{2\alpha_0 n(1-\alpha_0)^n}{L^2}\left(\frac{\mu_{d0}v_t}{g\rho_d}\right)\left[\frac{1-\left(\frac{L^2}{L_0^2}\right)}{N(\tilde{c}_{0+} + \frac{2}{3}N)}\right]^{1/2}. \qquad [21]$$

We choose $2a|M_0|$ because it is the amplitude of the wave (see [22]). For the case $L > L_0$ an equilibrium amplitude exists only if $N(\tilde{c}_{0+} + \frac{2}{3}N) < 0$. However, in our model, N is positive ($\tilde{c}_{0+} > 0$ in both our model and Liu's model) and there is no equilibration. Therefore, if $L > L_0$ an initial disturbance will not equilibrate, and if $L < L_0$ the disturbance will decay to zero (as predicted by linear theory) only when the initial amplitude is less than the equilibrium amplitude given by [21]. If the amplitude is larger than this critical amplitude, the disturbance will also grow (see figure 1). The solid curve is the graph of [21]. The dashed curve is based on the numercal calculations of the next section and shows agreement with the nonlinear theory for small amplitudes but disagreement for large amplitudes.

With the change in notation taken into account, [21] agrees to the order considered with Liu's amplitude equation 5.4 if $(\tilde{c}_{0+} + \frac{2}{3}N)$ is replaced by $\tilde{c}_{0+}$. Since Liu reasons (in the current notation) that N is negative, he concludes using his formula that for the supercritical case ($L > L_0$) an equilibrium amplitude always exists for $t \to \infty$. This is opposite to our conclusions and, as discussed earlier, unreasonable in view of the interpretations of the effects of the nonlinearities. The numerical calculations in the next section also show that there can be no equilibration for $L > L_0$ in the small amplitude regime because there are no small amplitude periodic solutions for these values of L.

It is clear from [17b, c] and [18b] that the modes $\exp[i(2x^* + 8\tilde{t}_0)]$, $\exp[i(3x^* + 27\tilde{t}_0)]$, $\exp[i(3x^* + 9\tilde{t}_0)]$ and $\exp[i(x^* + 7\tilde{t}_0)]$ decay to zero as $t^* \to \infty$. Therefore, [29] has the (unstable) periodic traveling wave solutions:

$$a\alpha = 2a|M_0|\cos\left\{x^* + t^*R_2^{-1}\left[1 - \frac{2N^2|M_0|^2\epsilon^2}{3} + O(\epsilon^4)\right] - \theta_0\right\} + O(a\epsilon). \qquad [22]$$

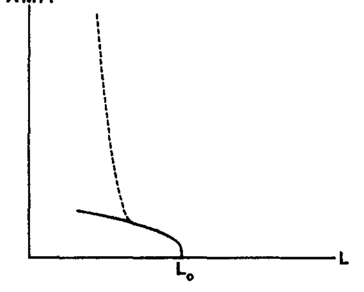

Figure 1. Graph of the equilibrium amplitude vs wavelength.

5. NUMERICAL CALCULATIONS

The periodic traveling wave solutions found in section 4 are restricted to very small amplitudes with $a \ll R_2^{-1} \ll 1$, even though [A.12] and [A.14] assume only $a \ll 1$ and $R_2^{-1} \ll 1$. The existence of periodic traveling wave solutions of [A.14] that includes the results of section 4 and extends it to amplitudes of order R_2^{-1} has been demonstrated both numerically (Christie & Ganser 1989) and analytically using perturbation theory (Ganser & Drew 1987). Both of these papers recover the results of section 4 when $a \ll R_2^{-1}$ and in particular the formula for the amplitude equation [21].

We calculate numerically, periodic traveling wave solutions of the original system given by [1a], [4] and [5]. This will yield further verification of the approximations and analysis used in this paper and also allow extension to cases when a and R_2^{-1} are not necessarily small and L is not large. The traveling wave solutions of [1a], [4] and [5] are functions of $\zeta = (x - st)/L$ which solve

$$-\frac{c}{v_t}\frac{1}{(L')^2}(1-\alpha)^n \frac{\partial\left(\dfrac{f(\alpha)}{\alpha}\dfrac{\partial\alpha}{\partial\zeta}\right)}{\partial\zeta} = \frac{1}{v_t}\left[\alpha v_d - \alpha j + v_t\alpha(1-\alpha)^n\right] - \frac{(1-\alpha)^n}{\alpha^2}\frac{L_c'}{L'}\left(\frac{c}{v_t} + \alpha\sqrt{F'}\right)\left(\frac{c}{v_t} - \alpha\sqrt{F'}\right)\frac{\partial\alpha}{\partial\zeta}, \qquad [23]$$

where

$$v_d = \frac{c}{\alpha} + s,$$

$$\mu_d = \mu_0 f(\alpha),$$

$$L_1^2 = \frac{\mu_0 v_t}{g\rho_d},$$

$$L' = \frac{L}{L_1}$$

and

$$L_c' = \frac{v_t^2}{gL_1}.$$

This equation contains the two parameters denoted by c, the integration constant from [1a], and s, the speed of the wave. It is preferable to change to the variables α_1 and α_2 $(0 \leqslant \alpha_2 \leqslant \alpha_1 \leqslant 1)$ which are the equilibrium points for [23] $[\alpha v_d - \alpha j + v_t\alpha(1-\alpha)^n = 0]$. This gives

$$-\frac{c}{v_t}\frac{1}{(L')^2}(1-\alpha)^n \frac{\partial\left(\dfrac{f(\alpha)}{\alpha}\dfrac{\partial\alpha}{\partial\zeta}\right)}{\partial\zeta} = \left(\frac{\alpha-\alpha_1}{\alpha_1-\alpha_2}\right)[\alpha_2(1-\alpha_2)^n - \alpha_1(1-\alpha_1)^n] + \alpha(1-\alpha)^n - \alpha_1(1-\alpha_1)^n - \frac{(1-\alpha)^n}{\alpha^2}\frac{L_c'}{L'}\left(\frac{c}{v_t} + \alpha\sqrt{F'}\right)\left(\frac{c}{v_t} - \alpha\sqrt{F'}\right)\frac{\partial\alpha}{\partial\zeta} \qquad [24]$$

with

$$\frac{c(\alpha_1,\alpha_2)}{v_t} = -\alpha_1(1-\alpha_1)^n - \alpha_1\left[\frac{\alpha_2(1-\alpha_2)^n - \alpha_1(1-\alpha_1)^n}{\alpha_1-\alpha_2}\right].$$

It can be shown (Fanucci *et al.* 1981; Needham & Merkin 1983) that the equilibrium point at $\alpha_2 = \bar{\alpha}_2$ is a center when

$$\frac{c(\alpha_1,\bar{\alpha}_2)}{v_t} = \bar{\alpha}_2\sqrt{F'}. \qquad [25]$$

Equation [25] has the solution $\alpha_1 = \bar{\alpha}_2 = \alpha_0^u$ (see [9b]). This solution continues with $\bar{\alpha}_2 < \alpha_0^u$ having a corresponding $\alpha_1 > \alpha_0^u$ that solves [25].

For a given $\alpha_0 < \alpha_0^u$ we are interested in the bifurcation at the critical point $L = L_0$ given by [10], or in terms of the current notation,

$$(L_0')^2 = \left(\frac{L_0}{L_1}\right)^2 = \frac{f(\alpha_0)(1-\alpha_0)^n\sqrt{F'}}{n\alpha_0(1-\alpha_0)^{n-1} - \sqrt{F'}}. \tag{26}$$

The values $L' = L_0'$, $\alpha_2 = \alpha_0$ and the corresponding α_1 solution of [25] with $\bar{\alpha}_2 = \alpha_0$, substituted into [24], gives the critical periodic solution of zero amplitude. Using the method of continuation, we calculate numerically nontrivial periodic solutions of [24] starting with these parameter values.

A value for L_c' is needed for the calculation. As stated in section 2, μ_0 is of the order of 10 g/(cm s). To agree with the calculations of Liu, we will let $f(\alpha) = \alpha^{-1}$. The terminal velocity v_t for particles with diameter of the order of a few hundred microns in a gas fluidized bed with particle density $\rho_d = O(3\text{ g/cm}^3)$ is 3–5 m/s. This gives $L_1 = O(1\text{ cm})$ and $L_c' = O(100)$. For the numerical calculations we choose $L_c' = 200$. For these parameter values, and $\alpha_0 \cong 0.6$, we have $L_c = O(1\text{ cm})$. Thus, with these parameter values, the analysis in section 4 requires $L \gg 1$ cm or $L' \gg 1$. Actually, we shall see that there is good agreement with the theory down to $L' = O(5)$.

The constant value for F' is

$$F' = n^2(1-\alpha_0^u)^{2n-2}(\alpha_0^u)^2. \tag{27}$$

This constant has been selected so that the critical state is α_0^u. We choose $\alpha_0^u = 0.6$ and $n = 3.5$ in the numerical calculations.

In the numerical method, [24] is discretized on a uniform grid of points by a second order finite difference method. Periodic boundary conditions are chosen as $\alpha(0) = \alpha(2\pi) = d_1$ and $\alpha'(0) = \alpha'(2\pi) = d_2$, where d_1 and d_2 are prescribed constants. These four conditions determine the solution $\alpha(x)$ of [24] and the unknowns α_1 and α_2. The system of algebraic equations produced by the method is solved by Newton's method with 3 or 4 iterations being required, typically, for the maximum norm of the difference between successive iterates to be $< 10^{-8}$. Periodic solutions are found starting with $d_1 = \alpha_0$, d_2 small and increasing d_2 in small increments. The initial values for each Newton step are taken to be those computed at the previous d_2 value.

The surface representing the two-parameter family of periodic traveling wave solutions found numerically is given in figure 2. This shows the graph of the numerical amplitude (one-half of the difference of the maximum and minimum numerical approximations of α), as a function of L', and α_0 which is the average value of the wave over one period. Recall that the wavelength in physical variables is $2\pi L$. To help visualize the surface, traces of the surface are generated for fixed values of L' starting from 1.5 in increments of 0.5. The numerical amplitude ranges in value from 0 to 0.1. The curve at the base of the surface (numerical amplitude = 0) is therefore the critical curve given by [26] from which the nontrivial periodic solutions bifurcate. The critical curve separates the linearly stable values of α_0 and L' (hatched region) from the linearly unstable values (unhatched region) in the AMP. = 0 plane. The curve Y in figure 2, parallel to the L' – AMP. plane, is the curve in figure 1 shown as part of the surface with $L_0' = 6.5$ and $\alpha_0 = 0.59920985$.

It is clear from the surface that, for large wavelengths, there are only small amplitude solutions with average value α_0 corresponding to linearly stable values of α_0 and L'. Consequently, small amplitude perturbations about an unstable α_0 state cannot equilibrate since there are no periodic solutions with the same average, α_0, or wave number as the perturbation. This agrees with the analysis in section 4. However, for L' sufficiently small there are such periodic solutions as indicated by the surface.

Table 1 compares the numerical amplitude of the solutions with the amplitude predicted by [21]. As should be expected, the best agreement is for large L' with L_0' close to L'. However, there is good agreement with L' as small as 5 and the amplitude as large as 0.005.

Three representative periodic solutions are shown in figure 3 where the broken line denotes the average value α_0. Their corresponding locations on the surface in figure 2 are indicated by the points A ($\alpha_0 = 0.598688$, $L' = 5$), B ($\alpha_0 = 0.660567$, $L' = 14$) and C ($\alpha_0 = 0.545393$, $L' = 1.5$). Solution A is of the type analyzed in section 4 with small amplitude (0.001064) and relatively long wavelength. Its Fourier series is dominated by its primary mode. On the other hand, solutions B and C contain important contributions from higher harmonics and the nonlinear analysis of section 4 is certainly not relevant to them. Solutions B and C are similar to those discussed in Ganser & Drew (1987),

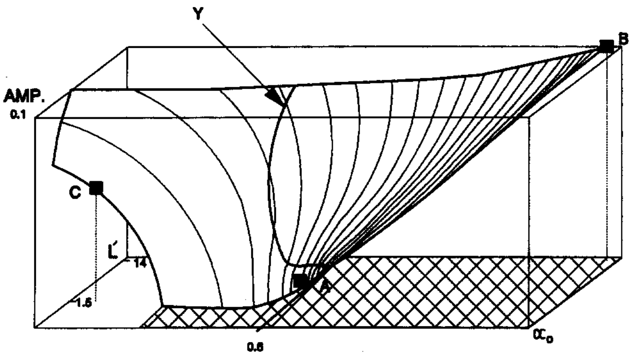

Figure 2. The surface representing the family of periodic solutions.

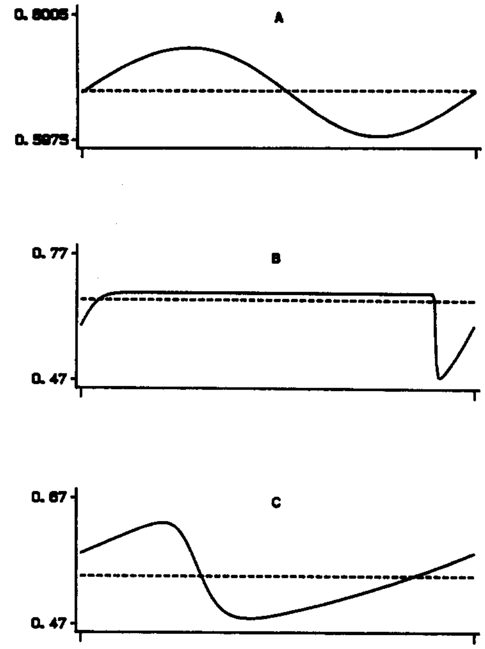

Figure 3. Three representative periodic solutions.

Table 1. Comparison between the numerical amplitude and the amplitude predicted by [21]

α_0	L'_0	L'	Numerical amplitude	$2a\|M_0\|$
0.599837	14.262425	14	1.5313E − 4	1.5113E − 4
0.599855	15.147080	14	3.0852E − 4	3.0205E − 4
0.599963	30.000707	14	7.4293E − 4	6.9869E − 4
0.599409	7.510654	7.5	1.5211E − 4	1.4777E − 4
0.599480	8.003345	7.5	1.0198E − 3	9.6769E − 4
0.599924	20.908399	7.5	2.8635E − 3	2.5709E − 3
0.598658	5.001370	5	1.5144E − 4	1.4777E − 4
0.599078	6.020123	5	4.0055E − 3	3.4951E − 3
0.599548	8.575045	5	6.0005E − 3	5.0628E − 3

where [A.14] is analyzed. In that paper, a class of traveling waves are constructed from a smooth transition solution which satisfies the equation with $\mu_{d0} = 0$ and a boundary layer solution. We are currently working on this analysis for [24].

6. CONCLUDING REMARKS

The analysis given in this paper demonstrates that for the larger (macroscopic) wavelengths there is no equilibration of small amplitude perturbations—at least in the small amplitude regime. However, figure 2 suggests that there may be equilibration for the smaller (microscopic) wavelengths at relatively small amplitudes. To show this, it would be necessary to analyze the stability of these solutions. If amplitudes >0.1 were included in figure 2, it would show that for the larger wavelengths the surface does fold back and begins to "cover" the linearly unstable (α_0, L') region as it already does for the smaller wavelengths. This was not shown because the results are nonphysical with unreasonably large particle concentrations. However, it does indicate that there are perhaps stable but necessarily large amplitude periodic solutions in a more realistic model.

Greater physical relevance would be achieved by the inclusion of a particle pressure $F(\alpha)$ which increases rapidly at a packing concentration. Such forms for $F(\alpha)$ have already been used by Fanucci *et al.* (1979, 1981). Similar forms are discussed in Drew (1983) and described there as approaching an incompressible model.

As discussed in section 5, with respect to solutions B and C in figure 3, these large amplitude solutions have smooth transitions separated by shock-like structures. This indicates that a model without viscosity and appropriate shock conditions yields similar results for large amplitudes.

It is interesting to speculate that the corresponding surface for liquid fluidized particles may fold at much smaller amplitudes. If this indeed is the case, and these solutions are stable, it would be an important step in the understanding of the mathematical model. In view of the previous remarks, we hypothesize that the instabilities so prevalent in the mathematical models are much more severe than presently believed. As the volumetric flux approaches the critical value and the bed becomes unstable, the model predicts relatively large amplitude disturbances rather than small bubbles or slugs. For gas fluidized particles, large amplitudes appear to be on the order of 0.1 or more. For liquid fluidized particles, "large amplitudes" are probably much smaller than 0.1. This picture is consistent with the well-known fact that the instability predicted by linear theory is much more severe, in general, for gas fluidized beds than for liquid fluidized beds (Anderson & Jackson 1968). We are currently studying the original [1a], [3a, b] and [5] to see if this description is valid for more realistic choices of F' and for liquid fluidized particles.

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APPENDIX

In this section we derive a small amplitude and long wavelength approximation to [1a], [4] and [5] for wave numbers near L_0^{-1} and α_0 states close to α_0^u. Studying wave numbers near L_0^{-1} yields solutions with small amplitude and studying only α_0 states near α_0^u implies that R_0^{-1} is small (see [10]).

Substitution of [7a–e] into [1a], [4] and [5] yields (superscript $\sim$ dropped)

$$\frac{\partial \alpha}{\partial t^*} - \frac{c_k}{c_{c0}} \frac{\partial \alpha}{\partial x^*} + \frac{\partial((\alpha_0 + a\alpha)v_d)}{\partial x^*} = 0, \quad \text{[A.1]}$$

$$(\alpha_0 + a\alpha)(v_d - v_c) - \alpha + v_c = 0 \quad \text{[A.2]}$$

and

$$-(\alpha_0 + a\alpha)(1 - \alpha_0 - a\alpha)\left(\frac{\partial v_d}{\partial t^*} - \frac{c_k}{v_{c0}}\frac{\partial v_d}{\partial x^*} + a v_d \frac{\partial v_d}{\partial x^*}\right)\frac{v_{c0}\rho_d}{LB_0} - (1 - \alpha_0 - a\alpha)\frac{\rho_d v_t F'}{L v_{c0} B_0}\frac{\partial \alpha}{\partial x^*}$$

$$-\frac{(1 - \alpha_0 - a\alpha)(\alpha_0 + a\alpha)}{(1 - \alpha_0)\alpha_0 a} + \frac{B}{aB_0}(1 + a v_c - a v_d) + \frac{\mu_d}{B_0 L^2}(1 - \alpha_0 - a\alpha)\frac{\partial\left((\alpha_0 + a\alpha)\frac{\partial v_d}{\partial x^*}\right)}{\partial x^*} = 0. \quad \text{[A.3]}$$

Since $a \ll 1$, a small amplitude expansion is performed keeping terms of $O(a)$ and omitting terms $O(a^2)$. A technical point is the retention of higher order dispersive terms. The basic dispersion in the system is of order R_2^{-1}. Since $R_2^{-1} \ll 1$ for the wavelengths of interest, terms of order $(R_2^{-1})^2$ and (aR_2^{-1}) will be omitted. Likewise, only diffusive-like terms of order (νR_2^{-1}) and (νa) will be kept while omitting quantities of order $\nu(R_2^{-1})^2$, $\nu a R_2^{-1}$, νa^2, $\nu^2 a$ and $\nu^2 R_2^{-1}$.

By using [A.2] in [A.3] to eliminate v_c it is possible to solve for v_d. Since we will ultimately substitute an expression for $(\alpha_0 + a\alpha)v_d$ into [A.1], it is convenient now to solve for $(\alpha_0 + a\alpha)v_d$:

$$(\alpha_0 + a\alpha)v_d = H[a\alpha] + \frac{(1 - \alpha_0 - a\alpha)^n}{(1 - \alpha_0)^n}\left[R_1^{-1}\frac{\partial\left((\alpha_0 + a\alpha)\frac{\partial v_d}{\partial x^*}\right)}{\partial x^*} + \nu\phi\right], \quad \text{[A.4]}$$

where

$$H[a\alpha] = \frac{(\alpha_0 + a\alpha)(1 - \alpha_0)^{1-n}}{a}[(1 - \alpha_0)^n - (1 - \alpha_0 - a\alpha)^n]$$

$$= \frac{c_k}{v_{c0}}\alpha - N a\alpha^2 + O(a^2),$$

$$N = n\left[\frac{(n-1)\alpha_0}{2(1 - \alpha_0)} - 1\right]$$

and

$$\phi = (\alpha_0 + a\alpha)\left[-\frac{\partial v_d}{\partial t^*} + \left(\frac{c_k}{v_{c0}} - a v_d\right)\frac{\partial v_d}{\partial x^*}\right] - \frac{v_t^2 F'}{v_{c0}^2}\frac{\partial \alpha}{\partial x^*}.$$

Using [A.4] it is possible to solve for v_d to the desired order. This calculation gives

$$v_d = \frac{c_k \alpha}{v_{c0}\alpha_0} - \left(\frac{c_k}{v_{c0}\alpha_0^2} + \frac{N}{\alpha_0}\right)a\alpha^2 + \frac{R_2^{-1}}{\alpha_0}\frac{\partial^2 \alpha}{\partial x^{*2}} + O(R, \nu\phi). \quad \text{[A.5]}$$

Here, R represents remainder terms of order a^2, aR_2^{-1} and $(R_2^{-1})^2$. Substitution of [A.5] into the expression for ϕ yields

$$\phi = -\frac{c_k}{v_{c0}}\frac{\partial \alpha}{\partial t^*} + \left[\left(\frac{c_k}{v_{c0}}\right)^2 - \frac{v_t^2 F'}{v_{c0}^2}\right]\frac{\partial \alpha}{\partial x^*} - \frac{2}{\alpha_0}\left(\frac{c_k}{v_{c0}}\right)^2 a\alpha\frac{\partial \alpha}{\partial x^*}$$

$$- 2N\left(\frac{c_k}{v_{c0}}\right)a\alpha\frac{\partial \alpha}{\partial x^*} + \left(\frac{c_k}{v_{c0}}\right)R_2^{-1}\frac{\partial^3 \alpha}{\partial x^{*3}} + O[R, \nu\phi, a\alpha_{t^*}, R_2^{-1}\alpha_{t^*}]. \quad \text{[A.6]}$$

It is useful to rewrite ϕ in terms of the characteristics of the corresponding hyperbolic problem ($\mu_d = 0$):

$$c_{\pm}[a\alpha] = -\frac{c_k}{v_{c0}} + av_d \pm \left(\frac{v_t^2 F'}{v_{c0}^2}\right)^{1/2}. \qquad [A.7]$$

Since $c_{0+} = O(R_2^{-1})$, it is possible with [A.5] to write

$$c_+[a\alpha] = c_{0+} + \tilde{c}_{0+} a\alpha + O(a^2, aR_2^{-1}, v\phi), \qquad [A.8]$$

where

$$\tilde{c}_{0+} = \frac{c_k}{v_{c0}\alpha_0} > 0$$

and

$$c_{0-} = -2\frac{c_k}{v_{c0}} + O(a, R_2^{-1}). \qquad [A.9]$$

Hence,

$$\phi = c_{0-}\left[\frac{1}{2}\frac{\partial \alpha}{\partial t^*} + (c_{0+} + \tilde{c}_{0+} a\alpha)\frac{\partial \alpha}{\partial x^*} + Na\alpha\frac{\partial \alpha}{\partial x^*} - \frac{1}{2}R_2^{-1}\frac{\partial_3 \alpha}{\partial x^{*3}}\right] + O(R, v\phi, a\alpha_{,t^*}, R_2^{-1}\alpha_{,t^*})$$
$$= O(\alpha_{,t^*}, a, R_2^{-1}). \qquad [A.10]$$

The expression for $(\alpha_0 + a\alpha)v_d$ can now be simplified with the help of [A.5] and [A.10] to

$$(\alpha_0 + a\alpha)v_d = \frac{c_k \alpha}{v_{c0}} - Na\alpha^2 + R_2^{-1}\frac{\partial^2 \alpha}{\partial x^{*2}}$$
$$+ vc_{0-}\left[\frac{1}{2}\frac{\partial \alpha}{\partial t^*} + (c_{0+} + \tilde{c}_{0+} a\alpha)\frac{\partial \alpha}{\partial x^*} + Na\alpha\frac{\partial \alpha}{\partial x^*} - \frac{1}{2}R_2^{-1}\frac{\partial^3 \alpha}{\partial x^{*3}}\right]$$
$$+ O(R, vR, v^2 a, v^2 R_2^{-1}, v^2\alpha_{,t^*}, va\alpha_{,t^*}, vR_2^{-1}\alpha_{,t^*}). \qquad [A.11]$$

Finally, substitution of [A.5] into [A.1] gives

$$\frac{\partial \alpha}{\partial t^*} - 2Na\alpha\frac{\partial \alpha}{\partial x^*} + R_2^{-1}\frac{\partial^3 \alpha}{\partial \alpha^{*3}} + vc_{0-}\frac{\partial\left((c_{0+} + \tilde{c}_{0+} a\alpha)\frac{\partial \alpha}{\partial x^*} + 2Na\alpha\frac{\partial \alpha}{\partial x^*} - R_2^{-1}\frac{\partial^3 \alpha}{\partial x^{*3}}\right)}{\partial x^*}$$
$$= O(R, vR, v^2\alpha, v^2 R_2^{-1}). \qquad [A.12]$$

We have used the approximation

$$\frac{\partial \alpha}{\partial t^*} = 2Na\alpha\frac{\partial \alpha}{\partial x^*} - R_2^{-1}\frac{\partial^3 \alpha}{\partial x^{*3}} + O(vR_2^{-1}, va) \qquad [A.13]$$

to estimate the time derivatives in [A.11]. The use of equation [A.13] also yields the following equivalent equation to the order considered:

$$\frac{\partial \alpha}{\partial t^*} - 2Na\alpha\frac{\partial \alpha}{\partial x^*} + R_2^{-1}\frac{\partial^3 \alpha}{\partial x^{*3}} + vc_{0-}\frac{\partial\left((c_{0+} + \tilde{c}_{0+} a\alpha)\frac{\partial \alpha}{\partial x^*} + \frac{\partial \alpha}{\partial t^*}\right)}{\partial x^*} = O(R, vR, v^2\alpha, v^2 R_2^{-1}). \qquad [A.14]$$

Note that because $\alpha_{,t^*} = O(R_2^{-1}, a)$, the $\alpha_{,t^*x^*x^*}$ and $\alpha_{,t^*t^*}$ terms in [8] are of higher order and do not appear here. Thus, in this approximation, we are beyond the boundary layer since $\alpha_{,t^*t^*}$ is small. Liu (1983) makes the same assumption later in his calculation (see his equation 5.1). Another consequence of omitting terms of order $(R_2^{-1})^2$ is that the formula for the critical wave number, [10], is modified to

$$-R_{20}^{-1} = c_{0+}. \qquad [A.15]$$